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Fattening complex manifolds: curvature and Kodaira–Spencer maps

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We present a calculus whereby the curvature of a geometry arising from any generalized twistor correspondence is related to an obstruction-theoretic classification of the infinitesimal neighborhoods of submanifolds of its twistor space. The crux of the argument involves a relation between Kodaira–Spencer maps and the Penrose transform.

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1. Introduction

Twistor correspondences are widely occurring phenomena whereby the differential geometry of one space is encoded in the complex structure of another. This may seem rather paradoxical, insofar as complex structures are, by their nature, locally trivial or, if you like, *flat*, whereas the occurrence of the term "differential" in the name "differential geometry" precisely stresses the fact that the latter subject is primarily concerned with the *curvature* of spaces. Whither has the curvature vanished, then, in this looking-glass world of twistor spaces? As Roger Penrose has taught us, the answer is that it is encoded not in the vicinity of a point, but rather in the vicinity of a compact complex submanifold. The purpose of this paper is to present a systematic calculus whereby one can decipher geometric information concerning curvature, torsion, and their derivatives from a twistorial encryption.

The basic object to be studied in this article is that of an *infinitesimal neighborhood* in this sense of Griffiths [6], meaning a complex ringed space which represents a jet of an embedding of one complex manifold in another.

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In order to clarify this notion, we define an abstract version of it, called a *fattened complex manifold*, and give an obstruction-theoretic classification of all fattenings of a given complex manifold*. In the context of various twistor correspondences, we then relate these obstructions to curvature via considerations involving Kodaira–Spencer maps, thereby putting a number of applications in the existing literature (e.g., refs. [3,10]) on a firm footing.

It is a pleasure to dedicate this article to Roger Penrose on the occasion of his 60th birthday. We have learned a great deal from his insights over many years.

2. The obstruction theory of fattenings

Let Y be a complex manifold, and let $X \subset Y$ be a closed submanifold. The mth-order infinitesimal neighborhood of X in Y is [6] the ringed space $X^{(m)} = (X, \mathcal{O}_{(m)})$, where

$$\mathcal{O}_{(m)} = (\mathcal{O}_Y / \mathcal{I}^{m+1})|_X.$$

Here, $\mathcal{O}_Y$ is the sheaf of holomorphic functions on Y and $\mathcal{I} \subset \mathcal{O}_Y$ is the ideal sheaf of functions vanishing on X. These infinitesimal neighborhoods arise as the natural setting for formal power-series solutions of extension problems in which analytic objects on X are to be extended to a neighborhood of X in Y. One may use the above example as a guide in defining a *fattening* of a complex manifold $(X, \mathcal{O})$ of order m and codimension k to be a ringed space $X^{(m)} = (X, \mathcal{O}_{(m)})$, where $\mathcal{O}_{(m)}$ is locally isomorphic to

$$\mathcal{O}_{m,k} \equiv \mathcal{O}[\zeta^1, \ldots, \zeta^k] / (\zeta^1, \ldots, \zeta^k)^{m+1}$$

and is equipped with an augmentation homomorphism $\alpha : \mathcal{O}_{(m)} \to \mathcal{O}$ which, with respect to the above local isomorphisms, just "forgets" $\zeta^1, \ldots, \zeta^k$.

When $k = 1$, a fattening is the same as a *thickening* in the terminology of ref. [4]. The purpose of this section is to provide a solution to the following problem: given a fattening $X^{(m)}$ of X, when does there exist a fattening $X^{(m+p)}$ of higher order which extends it and, if so, how many extensions are there? The results given here will strengthen and generalize the result for thickenings given in ref. [4]. Let $X^{(m)}$ be a fattening of X, and observe that there is a natural collection $\{X^l \mid l = 0, 1, \ldots, m\}$ of associated fattenings obtained by setting

$$\mathcal{O}_{(l)} \equiv \mathcal{O}_{(m)} / \mathcal{I}_{(m)}^{l+1}$$

* As it turns out, this classification has a rather long prehistory; cf., e.g., ref. [12]. For a supersymmetric analogue, cf. refs. [4,16].

where $\mathcal{I}_{(m)} \subset \mathcal{O}_{(m)}$ is the ideal of nilpotents; one says that $X^{(m)}$ *extends* $X^{(l)}$. These lower order fattenings come equipped with a useful family of $\mathcal{O}_{(l)}$-modules, namely $\mathcal{I}^q_{(p)}$ for $p \leq l + q$. In particular, provided $m > 0$, one may define the *conormal bundle* $N^* \to X$ by $\mathcal{O}(N^*) = \mathcal{I}_{(1)}$, noting that the right hand sheaf is locally free over $\mathcal{O}_{(0)} = \mathcal{O}$. The *normal bundle* $N \to X$ is defined to be its dual. These definitions are dictated by the archetypal examples of the infinitesimal neighborhoods of $X \subset Y$. In the special case of a thickening, $\mathcal{I}_{(p)}$ turns out to be locally free on $X^{(p-1)}$ for $p \leq m$ so that one can sensibly define a vector bundle extension of N^* to $X^{(p-1)}$ by

$$\mathcal{O}_{(p-1)}(N^*) \equiv \mathcal{I}_{(p)} \quad \text{if} k = 1, p \leq m$$

and duals or tensor products of N^* may also be extended to $X^{(p-1)}$.

In a similar spirit, one may mimic $TY|_X$ by defining the *extended tangent bundle* $\hat{T} \to X$ by

$$\begin{aligned} \mathcal{O}(\hat{T}) &= \mathcal{D}\mathrm{er}(\mathcal{O}_{(1)}, \mathcal{O}) \\ &:= \left\{ D : \mathcal{O}_{(1)} \to \mathcal{O} \,\middle|\, \begin{matrix} D \text{ is } \mathbb{C}\text{-linear} \\ D(fg) = fDg + gDf \end{matrix} \right\}. \end{aligned}$$

This extended tangent bundle fits into an exact sequence $0 \to T \to \hat{T} \to N \to 0$, where T is the holomorphic tangent bundle on X. From this sequence $\mathcal{O}_{(1)}$ can be reconstructed [4], so that the family of all first order fattenings of X with normal bundle N is naturally identified with $H^1(X, \mathcal{O}(T \otimes N^*))$. One can extend this tangent bundle to $X^{(p-1)}$, for $p \leq m$, by

$$\mathcal{O}_{(p-1)}(\hat{T}) \equiv \mathcal{D}\mathrm{er}(\mathcal{O}_{(p)}, \mathcal{O}_{(p-1)}).$$

If $p \geq 0$, there is a natural restriction homomorphism $\pi : \mathcal{A}\mathrm{ut}(\mathcal{O}_{(m+p)}) \to \mathcal{A}\mathrm{ut}(\mathcal{O}_{(m)})$, where $\mathcal{A}\mathrm{ut}$ denotes ring automorphisms which preserve the augmentation. Set $\mathcal{A}\mathrm{ut}_{(m)}(\mathcal{O}_{(m+p)}) \equiv \ker \pi$.

Lemma 1. *Suppose that* $p \leq m$. *Then for any fattening* $X^{(m+p)} = (X, \mathcal{O}_{(m+p)})$ *there is a natural isomorphism*

$$\mathcal{A}\mathrm{ut}_{(m)}(\mathcal{O}_{(m+p)}) \cong \mathcal{D}\mathrm{er}(\mathcal{O}_{(p)}, \mathcal{I}^{m+1}_{(m+p)}).$$

Proof. First one constructs an isomorphism

$$\sigma : \mathcal{A}\mathrm{ut}_{(m)}(\mathcal{O}_{(m+p)}) \to \mathcal{D}\mathrm{er}(\mathcal{O}_{(m+p)}, \mathcal{I}^{m+1}_{(m+p)}), \ \sigma(\Phi) = \Phi - 1.$$

To see that this is well defined, suppose that Φ is an automorphism of $\mathcal{O}_{(m+p)}$ acting trivially on $\mathcal{O}_{(m)}$. Then $(\Phi - 1)\mathcal{O}_{(m+p)} \subset \mathcal{I}^{m+1}_{(m+p)}$ and

$$\begin{aligned}(\Phi - 1)(fg) &= \Phi(f)\Phi(g) - fg \\ &= f(\Phi - 1)(g) + g(\Phi - 1)(f) + [(\Phi - 1)(f)][(\Phi - 1)(g)] \\ &= f(\Phi - 1)(g) + g(\Phi - 1)(f)\end{aligned}$$

because $p \leq m$ implies that $[\mathcal{I}^{m+1}_{(m+p)}]^2 = 0$. Thus, $\sigma(\Phi)$ is indeed a derivation if $\Phi \in \mathcal{A}\mathrm{ut}_{(m)}(\mathcal{O}_{(m+p)})$. Conversely, if $D \in \mathcal{D}\mathrm{er}(\mathcal{O}_{(m+p)}, \mathcal{I}^{m+1}_{(m+p)})$,

$$\begin{aligned}(1 + D)(fg) &= fg + fDg + gDf \\ &= fg + fDg + gDf + [Df][Dg] \\ &= [(1 + D)(f)][(1 + D)(g)],\end{aligned}$$

so that $1 + D$ is an automorphism acting trivially on $\mathcal{O}_{(m)}$, and σ is bijective.

To see that σ is isomorphism of sheaves of groups, notice that if $D_j \in \mathcal{D}\mathrm{er}(\mathcal{O}_{(m+p)}, \mathcal{I}^{m+1}_{(m+p)})$, for $j = 1, 2$, then

$$D_j(\mathcal{I}^{p+1}_{(m+p)}) \subset \mathcal{I}^{p}_{(m+p)} D_j \mathcal{I}_{(m+p)} \subset \mathcal{I}^{m+p+1}_{(m+p)},$$

so that $p \leq m$ implies $D_1 D_2 = 0$ and

$$(1 + D_1)(1 + D_2) = 1 + (D_1 + D_2).$$

Hence, σ is indeed an isomorphism as claimed. Finally, notice that there is a natural isomorphism

$$\rho : \mathcal{D}\mathrm{er}(\mathcal{O}_{(m+p)}, \mathcal{I}^{m+1}_{(m+p)}) \to \mathcal{D}\mathrm{er}(\mathcal{O}_{(p)}, \mathcal{I}^{m+1}_{(m+p)})$$

because elements of $\mathcal{D}\mathrm{er}(\mathcal{O}_{(m+p)}, \mathcal{I}^{m+1}_{(m+p)})$ annihilate $\mathcal{I}^{p+1}_{(m+p)}$ by the above correspondence. This gives an isomorphism

$$\rho\sigma : \mathcal{A}\mathrm{ut}_{(m)}(\mathcal{O}_{(m+p)}) \cong \mathcal{D}\mathrm{er}(\mathcal{O}_{(p)}, \mathcal{I}^{m+1}_{(m+p)})$$

as claimed. □

The right hand sheaf would be more useful if it were constructed out of $\mathcal{O}_{(p)}$ alone. The next lemma shows that such a construction is indeed possible.

Lemma 2. *For any fattening $X^{(m+p)} = (X, \mathcal{O}_{(m+p)})$,*

$$\mathcal{I}^{m+1}_{(m+p)} \cong \odot^{m+1}_{\mathcal{O}_{(p-1)}} \mathcal{I}_{(p)}$$

as a sheaf of $\mathcal{O}_{(p-1)}$-modules, where $\odot$ denotes symmetric tensor product.

Proof. There is a natural surjective morphism

$$\mu : \odot^{m+1}_{\mathcal{O}_{(p-1)}} \mathcal{I}_{(p)} \to \mathcal{I}^{m+1}_{(m+p)}$$

given by

$$\mu\Big(\sum_J \gamma_J \bigodot_{j=0}^{m} \beta_{J_j}\Big) \equiv \sum_J \tilde{\gamma}_J \prod_{j=0}^{m} \tilde{\beta}_{J_j},$$

where $\beta_{J_j} \in \mathcal{I}_{(p)}$, $\gamma_J \in \mathcal{O}_{(p-1)}$, and $\tilde{\gamma}_J$, $\tilde{\beta}_{J_j} \in \mathcal{O}_{(m+p)}$ have projections to $\mathcal{O}_{(p-1)}$ and $\mathcal{O}_{(p)}$, respectively, equal to γ_J and β_{J_j}, respectively, but are otherwise arbitrary.

To show that μ is injective, notice that the question is local, so that one may take $\mathcal{O}_{(m+p)} = \mathcal{O}_{m+p,k}$ which shall be treated as an $\mathcal{O}$-module. Each element of $\odot^{m+1}_{\mathcal{O}_{p-1,k}} \mathcal{I}_{p,k}$ may be expressed as

$$\sum_{q=0}^{p-1} \sum_{1 \le i_0 \le \cdots \le i_{m+q} \le k} f_{i_0 \cdots i_{m+q}} \zeta_{i_0} \odot \cdots \odot \zeta_{i_{m-1}} \odot (\zeta_{i_m} \cdots \zeta_{i_{m+q}}),$$

where the coefficients $f_{i_0 \cdots i_{m+q}}$ are elements of $\mathcal{O}$. The image of this element under μ is

$$\sum_{q=0}^{p-1} \sum_{1 \le i_0 \le \cdots \le i_{m+q} \le k} f_{i_0 \cdots i_{m+q}} \zeta_{i_0} \cdots \zeta_{i_{m+q}},$$

which can only be zero if all the coefficients $f_{i_0 \cdots i_{m+q}}$ vanish, since the terms $\zeta_{i_0} \cdots \zeta_{i_{m+q}}$, for $q \le p$, are independent generators over $\mathcal{O}$. □

With these tools in hand, we now give the main result of this section.

Theorem 3. *Suppose that* $0 < p \le m$. *The obstruction to the existence of a fattening* $X^{(m+p)}$ *extending a given fattening* $X^{(m)} = (X, \mathcal{O}_{(m)})$ *is an element of* $H^2(X, \mathcal{D}\mathrm{er}(\mathcal{O}_{(p)}, \odot^{m+1}_{\mathcal{O}_{(p-1)}} \mathcal{I}_{(p)}))$. *If this obstruction vanishes, the family of all such fattenings is acted upon freely and transitively by* $H^1(X, \mathcal{D}\mathrm{er}(\mathcal{O}_{(p)}, \odot^{m+1}_{\mathcal{O}_{(p-1)}} \mathcal{I}_{(p)}))$.

Proof. The family of all fattenings of X of order $m+p$ and codimension k is the non-Abelian sheaf cohomology set $H^1(X, \mathcal{A}\mathrm{ut}(\mathcal{O}_{m+p,k}))$. By lemmata 1 and 2 there is a short exact sequence of sheaves of groups

$$\mathcal{D}\mathrm{er}(\mathcal{O}_{p,k}, \odot^{m+1}_{\mathcal{O}_{p-1,k}} \mathcal{I}_{p,k}) \overset{j}{\rightarrowtail} \mathcal{A}\mathrm{ut}(\mathcal{O}_{m+p,k}) \overset{\pi}{\twoheadrightarrow} \mathcal{A}\mathrm{ut}(\mathcal{O}_{m,k}),$$

where π is the natural restriction map and j is obtained by following $(\rho\sigma)^{-1}$ with the natural inclusion of $\mathcal{A}\mathrm{ut}_{(m)}(\mathcal{O}_{m+p,k})$ into $\mathcal{A}\mathrm{ut}(\mathcal{O}_{m+p,k})$.

Since $\mathcal{D}\mathrm{er}(\mathcal{O}_{p,k}, \odot^{m+1}_{\mathcal{O}_{p-1,k}} \mathcal{I}_{p,k})$ is Abelian, the obstruction theory of ref. [4] applies. Thus, if $t \in H^1(\mathcal{A}\mathrm{ut}(\mathcal{O}_{m,k}))$, the obstruction to t being in the image of π is an element of $H^2(X, t\,\mathcal{D}\mathrm{er}(\mathcal{O}_{p,k}, \odot^{m+1}_{\mathcal{O}_{p-1,k}} \mathcal{I}_{p,k})t^{-1})$. Here, the sheaf $t\,\mathcal{D}\mathrm{er}(\mathcal{O}_{p,k}, \odot^{m+1}_{\mathcal{O}_{p-1,k}} \mathcal{I}_{p,k})t^{-1}$ is made from local copies of $\mathcal{D}\mathrm{er}(\mathcal{O}_{p,k}, \odot^{m+1}_{\mathcal{O}_{p-1,k}} \mathcal{I}_{p,k})$ by using a Čech representation of t as the set of transition functions via the conjugation action of $\mathcal{A}\mathrm{ut}(\mathcal{O}_{m,k})$ on $\mathcal{A}\mathrm{ut}_{(m)}(\mathcal{O}_{m+p,k})$. But

$$t\,\mathcal{D}\mathrm{er}(\mathcal{O}_{p,k}, \odot^{m+1}_{\mathcal{O}_{p-1,k}} \mathcal{I}_{p,k})t^{-1} = \mathcal{D}\mathrm{er}(\mathcal{O}_{(p)}, \odot^{m+1}_{\mathcal{O}_{(p-1)}} \mathcal{I}_{(p)})$$

if t is a Čech cohomology element representing $\mathcal{O}_{(m)}$. This yields the desired result. □

Note that two extensions $\mathcal{O}_{(m+p)}$ and $\overline{\mathcal{O}}_{(m+p)}$ of $\mathcal{O}_{(m)}$ are considered to be equivalent in the above context iff there is an isomorphism between them which induces the *identity* on $\mathcal{O}_{(m)}$. This is of course a stronger requirement than merely demanding that they be isomorphic as extensions of $\mathcal{O}$. Specifically, the equivalence classes for the latter weaker notion of equivalence are instead parameterized by the orbits of an action of $\Gamma(X, \mathcal{A}\mathrm{ut}_{(0)}(\mathcal{O}_{(m)}))$ on $H^1(X, \mathcal{D}\mathrm{er}(\mathcal{O}_{(p)}, \odot^{m+1}_{\mathcal{O}_{(p-1)}} \mathcal{I}_{(p)}))$.

Part of this theorem, namely the H^1 clause, may be checked more directly. Indeed, given $X^{(m+p)}$, the family of all fattenings of order $m+p$ agreeing with $X^{(m)}$ is $H^1(\mathcal{A}\mathrm{ut}_{(m)}(\mathcal{O}_{(m+p)}))$. If $p \le m$, one may use lemmata 1 and 2 to rewrite this as $H^1(X, \mathcal{D}\mathrm{er}(\mathcal{O}_{(p)}, \odot^{m+1}_{\mathcal{O}_{(p-1)}} \mathcal{I}_{(p)}))$.

A somewhat different way of stating the same result involves noticing that $\mathcal{D}\mathrm{er}(\mathcal{O}_{(p)}, \odot^{m+1}_{\mathcal{O}_{(p-1)}} \mathcal{I}_{(p)})$ may be rewritten as

$$\mathcal{D}\mathrm{er}(\mathcal{O}_{(p)}, \mathcal{O}_{(p-1)}) \otimes \odot^{m+1} \mathcal{I}_{(p)} = \mathcal{O}_{(p-1)}(\hat{T}) \otimes_{\mathcal{O}_{(p-1)}} \odot^{m+1}_{\mathcal{O}_{(p-1)}} \mathcal{I}_{(p)},$$

since a derivation from $\mathcal{O}_{(p)}$ to an $\mathcal{O}_{(p-1)}$-module is determined precisely by arbitrarily chosen images for $\zeta^1, \ldots, \zeta^k, z^1, \ldots, z^n \in \mathcal{O}_{(p)}$, where $\alpha(z^1), \ldots, \alpha(z^n) \in \mathcal{O}$ form a local coordinate system. This is particularly appealing in the cases when either k or p is one, in which case $\mathcal{O}_{(p-1)}(N^*) \equiv \mathcal{I}_{(p)}$ is a vector bundle on $X^{(p-1)}$. When one inserts these changes, the following is obtained:

Corollary 4. *Suppose that either $p = 1$ or that $k = 1$, and let $m \ge p$. The obstruction to finding $X^{(m+p)}$ extending a given $X^{(m)}$ is an element of $H^2(X, \mathcal{O}_{(p-1)}(\hat{T} \otimes \odot^{m+1} N^*))$. If this obstruction vanishes, then the family of all such fattenings $X^{(m+p)}$ is parameterized by $H^1(X, \mathcal{O}_{(p-1)}(\hat{T} \otimes \odot^{m+1} N^*))$.* □

When both k and p are one, this is the result of ref. [4].

3. Relation to Kodaira–Spencer maps

Let $D \subset \mathbb{C}$ denote a disk about $t = 0$ in the complex t-plane, and let $\pi : \mathcal{Y} \to D$ be a submersive holomorphic map, henceforth referred to as a *family*. We will think of such a family as a collection of complex manifolds $Y_t := \pi^{-1}(t)$ depending "holomorphically" on the parameter $t \in D$, and so often refer to such a family as a *deformation* of Y_t . The *Kodaira–Spencer obstruction* [8] of the family π is the element $\mathbf{ks}$ of $H^1(\mathcal{Y}, \mathcal{O}(T_\pi Y))$ which is the obstruction to lifting the vector field $\mathrm{d}/\mathrm{d}t$ to $\mathcal{Y}$; here $T_\pi Y := \ker D\pi$ is the vertical tangent bundle of π. To be precise, this obstruction may be represented in Čech cohomology by covering $\mathcal{Y}$ with open sets $\mathcal{U}_\alpha$ on which there exist vector fields v_α with $D\pi(v_\alpha) = \mathrm{d}/\mathrm{d}t$, which is possible precisely because π is submersive; then $\mathbf{ks}$ is the image in cohomology of the Čech cocycle $\{v_\alpha - v_\beta\} \in C^1(\{\mathcal{U}_\alpha\}, \mathcal{O}(T_\pi Y))$.

The *Kodaira–Spencer map* $\mathbf{KS}$ of π is the closely related function which assigns to any vector $v \in T_t D$ an element of $H^1(Y_t, \mathcal{O}(TY_t))$; namely, it assigns to $\lambda\,\mathrm{d}/\mathrm{d}t|_t$ the restriction of $\lambda\mathbf{ks}$ to Y_t. Such a map can equally well be defined for a family over an arbitrary complex manifold B, and again is a linear functional on the tangent space of the base, with values in the first cohomology of the fibers with coefficients in holomorphic vector fields. For the analysis of *proper* families, it typically turns out [8] that the Kodaira–Spencer map contains all the information about the family encoded in the obstruction $\mathbf{ks}$. However, the families most relevant to twistor correspondences are almost never proper. Nonetheless, the same phenomenon occurs: the Kodaira–Spencer maps contain all the geometrically relevant information concerning the deformation. One of our chief goals in this article is to explain why this is so.

Suppose now that, for some complex manifold X, we are given a closed inclusion $X \times D \hookrightarrow \mathcal{Y}$ for which π agrees with projection to the second factor. Along $X \times D$, we may then choose our local lifts v_α of $\mathrm{d}/\mathrm{d}t$ to agree with the product lift. This gives rise to a *relative* version $\hat{\mathbf{ks}} = \mathbf{ks}_{(0)} \in H^1(\mathcal{Y}, \mathcal{I}_X(T_\pi Y))$ of the Kodaira–Spencer obstruction, where $\mathcal{I}_X$ is the ideal sheaf of the closed subvariety $X \times D \subset \mathcal{Y}$. The previous obstruction $\mathbf{ks}$ is then nothing but the image of $\mathbf{ks}_{(0)}$ via the canonical map $j_* : H^1(\mathcal{Y}, \mathcal{I}_X(T_\pi Y)) \to H^1(\mathcal{Y}, \mathcal{O}(T_\pi Y))$. The relative Kodaira–Spencer map $\mathbf{KS}_{(0)}$ is defined in analogy with the previous case, but now takes its values in $H^1(Y_t, \mathcal{I}_X(TY_t))$.

In the same spirit, suppose that, in addition to a closed inclusion $X \times D \hookrightarrow \mathcal{Y}$, we are given an isomorphism between the mth infinitesimal neighborhood of $X \times D \subset \mathcal{Y}$ and that of $X \times D \subset Y_0 \times D$. Then we can further refine our choice of local liftings of $\mathrm{d}/\mathrm{d}t$, and this will result in a more refined version $\mathbf{ks}_{(m)}$ of the relative Kodaira–Spencer obstruction, which will now live in $H^1(\mathcal{Y}, \mathcal{I}_X^{m+1}(T_\pi Y))$. We also can then define a refined Kodaira–

Spencer map $\mathbf{KS}_{(m)}$ taking values in $H^1(Y_t, \mathcal{I}_X^{m+1}(TY_t))$. As we decrease m, these obstructions are interrelated by the obvious maps $H^1(\mathcal{Y}, \mathcal{I}_X^{m+1}(T_\pi Y)) \to H^1(\mathcal{Y}, \mathcal{I}_X^m(T_\pi Y))$ and $H^1(Y_t, \mathcal{I}_X^{m+1}(TY_t)) \to H^1(Y_t, \mathcal{I}_X^m(TY_t))$.

Now notice that every relative family in the above sense gives rise to a one-parameter collection of infinitesimal neighborhoods $X_t^{(m)}$ of $X \subset Y_t$. In light of the analysis in section 2, one should ask how the infinitesimal neighborhoods are affected by the change. The answer is delightfully simple. Suppose we have a family $\mathcal{Y}$ which is trivialized along X to order m. Then for each value of t and each $p \leq m$, the image of $\mathbf{KS}_{(m)}$ via the restriction map

$$H^1(Y_t, \mathcal{I}_X^{m+1}(TY_t)) \to H^1(X, \mathcal{I}_X^{m+1}(TY_t)/(\mathcal{I}_X^{m+p+1}(TY_t)))$$

yields a holomorphic one-form $\mathbf{KS}_{m,m+p}$ on D taking values in the vector space $H^1(X, \mathcal{D}\mathrm{er}(\mathcal{O}_{(p)}, \odot^{m+1}_{\mathcal{O}_{(p-1)}} \mathcal{I}_{(p)}))$. However, $\mathbf{KS}_{m,m+p}(\mathrm{d}/\mathrm{d}t)$ is exactly the t-derivative of the the element of $H^1(X, \mathcal{D}\mathrm{er}(\mathcal{O}_{(p)}, \odot^{m+1}_{\mathcal{O}_{(p-1)}} \mathcal{I}_{(p)}))$ which moves $X_0^{(m)}$ to $X_t^{(m)}$. Thus, for $a, b \in D$, $X_b^{(m)}$ is obtained from $X_a^{(m)}$ by applying

$$\int_a^b \mathbf{KS}_{m,m+p} \in H^1(X, \mathcal{D}\mathrm{er}(\mathcal{O}_{(p)}, \odot^{m+1}_{\mathcal{O}_{(p-1)}} \mathcal{I}_{(p)})) ,$$

where the above integral of a closed vector-space-valued one-form may be performed over any arc joining a to b in D. This then typically results in a reduction of the problem of geometrically interpreting the freedom occurring in theorem 3 to a linear calculation.

4. Deformations of leaf spaces

We now come to the crucial step in all twistor constructions, which is the construction of leaf spaces. Suppose that W is a complex manifold, and let $E \subset TW$ be an involutive holomorphic subbundle of its tangent bundle—i.e. an integrable holomorphic distribution of planes. Let $X \subset W$ be a complex submanifold such that $E \cap TX = 0$. There is a holomorphic foliation tangent to E, and we will assume that the leaves are simply connected and that the space of leaves W/E is a complex manifold; i.e., we assume that there exists a complex manifold Y and a holomorphic submersion $\mu : W \to Y$ such that E is the vertical tangent bundle $T_\mu W$ of μ. Assume, moreover, that the immersion $\mu|_X$ is injective. For our purposes, these added assumptions will impose no loss of generality, as there is always a neighborhood U of X for which the foliation has all these properties.

Let us now consider a one-complex-parameter family of such involutive subbundles $E_t \subset W$ with $E = E_0$ satisfying $E \cap TX = 0$. Then, perhaps by allowing t to range only over a neighborhood D of the origin, and by perhaps restricting ourselves to a neighborhood of X, we may assume that the union of the leaf spaces W/E_t give us family $\pi : \mathcal{Y} \to D$; moreover, this family contains $X \times D$ in a canonical way. What are the Kodaira–Spencer maps of such a family?

Before we even begin to answer this question, we must ask ourselves how best to represent the "derivative" of the family of foliations tangent to E_t. The formal tangent space at E of the space of holomorphic subbundles of TW is just

$$\Gamma(W, \mathcal{O}(\mathcal{H}\mathrm{om}(E, TW/E))) = \Gamma(W, \Omega^1_\mu(\mu^* TY)) .$$

Here $\Omega^1_\mu = E^*$ denotes the vertical cotangent bundle of the map $\mu = \mu_0$. Explicitly, the μ^*TY-valued relative one-form φ which represents $\mathrm{d}E_t/\mathrm{d}t$ at $t = 0$ is characterized by

$$\varphi(\xi(0)) = \mu_* \left(\frac{\mathrm{d}\xi}{\mathrm{d}t}(0) \right)$$

for any t-dependent vector field $\xi(t)$ which lies in in E_t. By assumption, however, the bundles E_t are all involutive, i.e., closed under Lie brackets. This then implies that the section φ of $\Omega^1_\mu(\mu^*TY)$ representing the derivative of E_t is in the kernel of the relative exterior derivative

$$d_\mu : \Omega^1_\mu(\mu^*TY) \to \Omega^2_\mu(\mu^*TY)$$

obtained by differentiating up the fibers of μ. Indeed, let $\xi(t),\ \eta(t) \in \mathcal{O}(E_t)$ be t-dependent holomorphic vector fields in on some open subset of W, and let ω be any holomorphic one-form on the corresponding region of $Y = Y_0$. Then

$$\begin{aligned}
&\langle \mu^*\omega\ ,\ d\varphi(\xi(0), \eta(0)) \rangle \\
&= \left[\xi\mu^*\omega\left(\frac{\mathrm{d}\eta}{\mathrm{d}t}\right) - \eta\mu^*\omega\left(\frac{\mathrm{d}\xi}{\mathrm{d}t}\right) - \mu^*\omega\left(\frac{\mathrm{d}}{\mathrm{d}t}[\xi, \eta]\right)\right]\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}[\xi\mu^*\omega(\eta) - \eta\mu^*\omega(\xi) - \mu^*\omega([\xi, \eta])]|_{t=0} \\
&\quad - \frac{\mathrm{d}\xi}{\mathrm{d}t}\mu^*\omega(\eta)|_{t=0} + \frac{\mathrm{d}\eta}{\mathrm{d}t}\mu^*\omega(\xi)|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}[d\omega(\mu_*\xi(t), \mu_*\eta(t))]|_{t=0} \\
&\quad - \left(\frac{\mathrm{d}\xi}{\mathrm{d}t}(0)\right)\mu^*\omega(\eta(0)) + \left(\frac{\mathrm{d}\eta}{\mathrm{d}t}(0)\right)\mu^*\omega(\xi(0)) \\
&= \left[\mathrm{d}\omega\left(\mu_*\frac{\mathrm{d}\xi}{\mathrm{d}t}(0), \mu_*\eta(0)\right) + \mathrm{d}\omega\left(\mu_*\xi(0), \mu_*\frac{\mathrm{d}\eta}{\mathrm{d}t}(0)\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&-\left(\frac{\mathrm{d}\xi}{\mathrm{d}t}(0)\right)0 + \left(\frac{\mathrm{d}\eta}{\mathrm{d}t}(0)\right)0 \\
= &\ 0 .
\end{aligned}
$$

We are thus entitled to think of φ as an element of $\Gamma(W,\mathcal{K})$, where $\mathcal{K}$ is defined by the exact sequence

$$0 \to \mathcal{K} \to \Omega^1_\mu(\mu^* TY) \xrightarrow{d_\mu} \Omega^2_\mu(\mu^* TY) \xrightarrow{d_\mu} \cdots$$

and so also fits into the exact sequence

$$0 \to \mu^{-1}\mathcal{O}(TY) \to \mathcal{O}(\mu^* TY) \xrightarrow{d_\mu} \mathcal{K} \to 0 .$$

Since the Kodaira–Spencer map $\mathbf{KS}(\pi,t)$ is concerned only with individual values of t, it suffices to evaluate it at $t = 0$. To do this, cover $Y = Y_0 := \pi^{-1}(0)$ with open sets $\mathcal{U}_\alpha$ over which π admits sections $\sigma_\alpha : \mathcal{U}_\alpha \to W$. For small values of t, we may assume, by shrinking our covering and if necessary our neighborhood of X, that E_t is everywhere transverse to the submanifolds $\sigma_\alpha(\mathcal{U}_\alpha) \subset W$. Also shrinking our neighborhood $D \subset \mathbb{C}$ of 0 as necessary, the total space $\mathcal{Y}$ of the family is covered by $\{\mathcal{V}_\alpha := \Phi_\alpha[\mathcal{U}_\alpha \times D]\}$, where

$$\Phi_\alpha : \mathcal{U}_\alpha \times D \to \mathcal{Y},\ (y,t) \mapsto (\mu_t \sigma_\alpha(y), t) .$$

We may then define a Čech cocycle representing $\mathbf{ks}(\pi)$ relative to the covering $\{\mathcal{V}_\alpha\}$ by $\{v_\alpha - v_\beta\}$, where $v_\alpha := \Phi_{\alpha *}(\mathrm{d}/\mathrm{d}t)$. Now, for some point $y \in \mathcal{Y}$, suppose that we let $c_t : [0,1] \to W$ be a family of smooth curves which are tangent to E_t and which join $\sigma_\alpha(y)$ to a point in the image of σ_β. Then

$$
\begin{aligned}
(v_\alpha - v_\beta)_y|_{t=0} &= [\Phi_{\alpha * \Phi_\alpha^{-1}(y)} - \Phi_{\beta * \Phi_\beta^{-1}(y)}]\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\Big|_{t=0} \\
&= \Phi_{\beta * \Phi_\beta^{-1}(y)} \frac{\mathrm{d}}{\mathrm{d}t}[\Phi_\beta^{-1}\Phi_\alpha]|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}[(\mu_t\sigma_\beta)^{-1}\mu_t\sigma_\alpha(y)]|_{t=0} \\
&= \mu_{0*}\left(\frac{\mathrm{d}}{\mathrm{d}t}c_t(1)\right)\Big|_{t=0} \\
&= \int_0^1 \mu_{0*}\left(\frac{\partial^2 c(t,s)}{\partial t\,\partial s}\right)\Big|_{t=0} \mathrm{d}s \\
&= \int_0^1 \varphi\left(\frac{\mathrm{d}c_t(s)}{\mathrm{d}s}\right)\mathrm{d}s \\
&= \int_{c_0} \varphi .
\end{aligned}
$$

Hence the the image of $\mathrm{d}/\mathrm{d}t|_{t=0}$ via the Kodaira–Spencer map $\mathbf{KS}(\pi)$ can be represented by the cocycle

$$\left\{ \int_{\sigma_\alpha}^{\sigma_\beta} \varphi \right\}_{\alpha\beta} ,$$

where the integral is understood to be performed along arbitrary paths in the fibers of μ; notice that, assuming the leaves of E are simply connected, the fact that this is well defined exactly amounts to the fact that φ is d_μ-closed.

Now consider the long exact sequence

$$\cdots \to \Gamma(\mathcal{O}(\mu^* TY)) \xrightarrow{d_\mu} \Gamma(\mathcal{K}) \xrightarrow{\delta} H^1(\mu^{-1}\mathcal{O}(TY)) \to \cdots$$

induced by the short exact sequence

$$0 \to \mu^{-1}\mathcal{O}(TY) \to \mathcal{O}(\mu^* TY) \xrightarrow{d_\mu} \mathcal{K} \to 0 . \tag{1}$$

The connecting homomorphism

$$\Gamma(\mathcal{K}) \xrightarrow{\delta} H^1(\mu^{-1}\mathcal{O}(TY))$$

may be constructed in Čech cohomology by taking differences of local splittings of the previous map

$$\Gamma(\mathcal{O}(\mu^* TY)) \xrightarrow{d_\mu} \Gamma(\mathcal{K})$$

on the double overlaps of an open cover. But assuming that the fibers of μ are simply connected, such a local splitting s_α may be constructed on $\mu^{-1}(\mathcal{U}_\alpha)$ from a local section σ_α by the prescription

$$[s_\alpha(\phi)](w) := \int_{\sigma_\alpha}^{w} \phi ,$$

where the integral is understood to be taken along arbitrary paths in the fibers of μ. Thus the connecting homomorphism is represented in Čech cohomology by

$$\phi \mapsto \left\{ \int_{\sigma_\alpha}^{\sigma_\beta} \phi \right\}_{\alpha\beta} .$$

But, applied to a relative one-form φ which represents the derivative of a family of foliations, this is, of course, precisely our formula for the Kodaira–Spencer map. Since a theorem of Buchdahl [2] shows that $\mu^* : H^1(Y, \mathcal{O}) \to H^1(W, \mu^{-1}\mathcal{O})$ is an isomorphism provided that the fibers of μ are simply connected, we have therefore proved the following:

Theorem 5. *Let $\mu : W \to Y$ be a holomorphic submersion with simply connected fibers. Then the Kodaira–Spencer map from linearized foliations to $H^1(Y, \mathcal{O}(TY))$ is given by the connecting homomorphism*

$$\delta : H^0(W, \mathcal{K}) \to H^1(W, \mu^{-1}\mathcal{O}(TY))$$

induced by the short exact sequence (1).

Now the submanifold $X \subset W$ projects isomorphically to a submanifold $X \subset Y_t$ for each value of our deformation parameter, and we therefore also have a relative Kodaira–Spencer map $\hat{\mathbf{KS}}$ of the family $\mathcal{Y}$ relative to X. This gives us a lifting $\hat{\delta}$

$$\begin{array}{ccc} & \overset{\hat{\delta}}{\nearrow} & H^1(Y, \mathcal{I}_X(TY)) \\ & & \downarrow j_* \\ H^0(W, \mathcal{K}) & \xrightarrow[\delta]{} & H^1(Y, \mathcal{O}(TY)) \end{array}$$

of the previous map δ; here j_* denotes the map induced by the inclusion homomorphism $j : \mathcal{I}_X(TY) \to \mathcal{O}(TY)$. But this lifting may again be realized on the level of Čech cohomology by the formula $\int_{\sigma_\alpha}^{\sigma_\beta} \varphi$, provided that we now take our sections σ_α to all restrict to $X \subset Y$ as the identity map $X \to X \subset W$.

Now $H^1(Y, \mathcal{I}_X(TY))$ and $H^1(Y, \mathcal{O}(TY))$ are related by the exact sequence

$$H^0(Y, \mathcal{O}(TY)) \to H^0(X, \mathcal{O}(TY)) \xrightarrow{\delta'} H^1(Y, \mathcal{I}_X(TY)) \xrightarrow{j_*} H^1(Y, \mathcal{O}(TY)) .$$

Thus $\hat{\delta}$ induces a map

$$f : H^0(W, \mathcal{O}(\mu^* TY))/H^0(Y, \mathcal{O}(TY)) \to H^0(X, \mathcal{O}(TY))/H^0(Y, \mathcal{O}(TY))$$

from the kernel of δ to the kernel of j_*, uniquely specified by the requirement that $\delta' f = \hat{\delta} d_\mu$. But the restriction map $\rho : H^0(W, \mathcal{O}(\mu^* TY)) \to H^0(X, \mathcal{O}(TY))$ has the property that $\delta' \rho = -\hat{\delta} d_\mu$, since δ' is obtained in Čech terms by taking the differences of pairs of local extensions to Y of a given section of $\mathcal{O}(TY)$ on $X \subset Y$, whereas, in light of our assumption regarding $\sigma_\alpha|_X$, such local extensions to Y of $\rho(g)$, $g \in \Gamma(W, \mathcal{O}(\mu^* TY))$, may be explicitly given in the form $g\sigma_\alpha$; thus

$$\delta' \rho(g) = \left[\{ g\sigma_\alpha - g\sigma_\beta \}_{\alpha\beta} \right]$$

$$= \left[\left\{-\int_{\sigma_\alpha}^{\sigma_\beta} d_\mu g\right\}_{\alpha\beta}\right]$$
$$= -\hat{\delta} d_\mu(g)\ .$$

Thus f is just the map induced by $-\rho$, and so has kernel

$$\frac{\Gamma(W,\mathcal{I}_X(\mu^*TY))}{[\Gamma(W,\mu^{-1}\mathcal{O}(TY))\cap\Gamma(W,\mathcal{I}_X(\mu^*TY))]} = \frac{\Gamma(W,\mathcal{I}_X(\mu^*TY))}{\Gamma(Y,\mathcal{I}_X(TY))}\ .$$

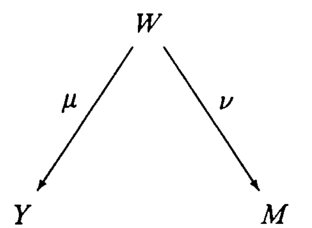
In summary:

Theorem 6. *Let $\mu : W \to Y$ be as before. Then the relative Kodaira–Spencer map $\hat{\mathbf{KS}} = \hat{\delta}$ from linearized foliations to $H^1(Y,\mathcal{I}_X(TY))$ fits into an exact sequence*

$$0 \to H^0(Y,\mathcal{I}_X(TY)) \to H^0(W,\mathcal{I}_X(\mu^*TY)) \xrightarrow{d_\mu} H^0(W,\mathcal{K}) \xrightarrow{\hat{\delta}} H^1(Y,\mathcal{I}_X(TY)).$$

5. Curvature and the Penrose transform

The *Penrose transform* is a machine whereby one relates analytic objects on one space to the solutions of differential equations on another. The basic setting is that of a *double fibration*



for which it is assumed that μ and ν are holomorphic submersions, $ker\ \mu_* \cap ker\ \nu_* = 0$, ν is proper, and the fibers of μ are one-connected. Analytic objects on Y may then be analyzed in terms of M by pulling back via μ and then pushing down via ν.

For our purposes, it will always be assumed that this double fibration arises as a complete analytic family in the sense of Kodaira [7], roughly meaning that it represents an open set in the space of compact complex submanifolds of Y. To be more precise, we assume that, for each fiber $X = \nu^{-1}(x)$ of ν, the map $\mu|_X$ is injective, and that the normal bundle N of its image satisfies the conditions

$$H^1(X, \mathcal{O}(N)) = 0, \tag{2}$$

$$\nu_*^0 \mu^* : T_x M \xrightarrow{\cong} H^0(X, \mathcal{O}(N)). \tag{3}$$

(The condition that μ be a submersion then implies $H^1(X, \mathcal{I}_w(N)) = 0$ for all $w \in W$, where $X = \nu^{-1}(\nu(w))$ and $\mathcal{I}_w \subset \mathcal{O}_X$ is the ideal sheaf of w.) In addition, we will assume that the fibers of ν are all *rigid*, in the sense that any fiber $X = \nu^{-1}(x)$ satisfies

$$H^1(X, \mathcal{O}(TX)) = 0 ; \tag{4}$$

by Kodaira–Spencer theory [8], it then follows that $W \xrightarrow{\nu} M$ is actually a locally trivial holomorphic X-bundle. By shrinking our domain, we may assume, without loss of generality, that M is a Stein manifold.

Let us apply the Penrose transform to $H^1(Y, \mathcal{O}(TY))$. The pull-back part of the process we have already encountered in the last section; namely, we have $H^1(Y, \mathcal{O}(TY)) = H^1(W, \mu^{-1}\mathcal{O}(TY))$, and the latter fits into an exact sequence

$$\cdots \to H^0(\mathcal{O}(\mu^* TY)) \xrightarrow{d_\mu} H^0(\mathcal{K}) \xrightarrow{\delta} H^1(\mu^{-1}\mathcal{O}(TY)) \to H^1(\mathcal{O}(\mu^* TY)) \to \cdots .$$

However, our completeness assumption (2) and the rigidity assumption imply that that $H^1(X, \mathcal{O}(TY)) = 0$ for each $X = \nu^{-1}(x)$, so that, using the fact that M is Stein, the Leray spectral sequence of ν tells us that $H^1(W, \mathcal{O}(\mu^* TY)) = 0$. Thus the Bockstein operator

$$\Gamma(\mathcal{K}) \xrightarrow{\delta} H^1(\mu^{-1}\mathcal{O}(TY))$$

is surjective, which in light of theorem 5 becomes the statement that all formal first order deformations of Y arise from formal first order deformations of the foliation μ. Moreover, completeness assumption (3) implies that $\Gamma(W, \mathcal{O}(\mu^* TY)) = \Gamma(W, \mathcal{O}(TW))$, so that we have an exact sequence

$$0 \to \Gamma(Y, \mathcal{O}(TY)) \to \Gamma(W, \mathcal{O}(TW)) \to \Gamma(W, \mathcal{K}) \to H^1(Y, \mathcal{O}(TY)) \to 0 .$$

Let us now recall that we also have an exact sequence

$$0 \to \mathcal{K} \to \Omega^1_\mu(\mu^* TY) \xrightarrow{d_\mu} \Omega^2_\mu(\mu^* TY) \xrightarrow{d_\mu} \cdots$$

and hence an exact sequence

$$0 \to \Gamma(\mathcal{K}) \to \Gamma(\Omega^1_\mu(\mu^* TY)) \xrightarrow{d_\mu} \Gamma(\Omega^2_\mu(\mu^* TY)) .$$

Thus we may also think of $H^1(Y, \mathcal{O}(TY))$ as the cohomology of the complex

$$\cdots \to \Gamma(W, \mathcal{O}(TW)) \xrightarrow{d_\mu} \Gamma(W, \Omega^1_\mu(\mu^* TY)) \xrightarrow{d_\mu} \Gamma(W, \Omega^2_\mu(\mu^* TY)) \to \cdots$$

corresponding to the fact that, with the stated hypotheses, the resolution

$$0 \to \mu^{-1}\mathcal{O}(TY) \to \mathcal{O}(\mu^* TY) \xrightarrow{d_\mu} \Omega^1_\mu(\mu^* TY) \xrightarrow{d_\mu} \Omega^2_\mu(\mu^* TY) \xrightarrow{d_\mu} \cdots$$

is acyclic in the relevant range. To complete the Penrose transform, we push this down to M. Namely, there are vector bundles $\mathcal{E}^p$ on M, with typical fiber $\Gamma(X, \mathcal{O}(\mu^* TY \otimes \bigwedge^p E^*))$, and $H^1(Y, \mathcal{O}(TY))$ becomes the cohomology of

$$\Gamma(M, \mathcal{E}^0) \xrightarrow{D_\mu} \Gamma(M, \mathcal{E}^1) \xrightarrow{D_\mu} \Gamma(M, \mathcal{E}^2),$$

where the operators D_μ are induced by the relative exterior derivatives d_μ. (In order to give this a good geometric interpretation, we will of course need more information concerning our particular twistor correspondence.)

6. Half conformally flat four-manifolds

We shall apply these results to Penrose's twistor theory of an anti-self-dual conformal spacetime. The standard theory [13,15] is as follows. Suppose M is a complex Riemannian manifold [9]. Then, the Riemannian curvature decomposes as follows [14]:

$$R_{abcd} = C_{abcd} + E_{abcd} + 2\Lambda g_{abcd},$$

where C_{abcd} is the Weyl curvature, E_{abcd} is equivalent to the trace-free Ricci curvature, and Λ is proportional to the scalar curvature. The Weyl curvature is conformally invariant in the sense that it is unchanged when the metric g_{ab} is replaced by αg_{ab} for any nowhere vanishing holomorphic function α. Furthermore, if M is four-dimensional then the Weyl curvature decomposes further into its *self-dual* and *anti-self-dual* parts $C_{abcd} = C^+_{abcd} + C^-_{abcd}$. In two-spinor notation [14], this reads

$$C_{abcd} = \tilde{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} + \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'}.$$

The non-linear graviton construction applies when M is anti-self-dual, i.e. when the self-dual part of the Weyl curvature vanishes. The metric itself is unnecessary for this construction. Only the metric up to scale (i.e. only a *conformal structure*) is needed. The non-linear graviton construction applies locally. Thus, suppose M has an anti-self-dual conformal structure and is also *geodesically convex* as in ref. [9]. Then, the *α-planes* in the tangent bundle to M integrate to give *α-surfaces* and hence the correspondence

$$\begin{array}{ccc} & F & \\ {\scriptstyle\mu}\swarrow & & \searrow{\scriptstyle\nu} \\ P & & M \end{array}$$

where P parameterizes the family of α-surfaces in M. In the conformally flat case $M = Gr_2(\mathbb{C}^4)$ and $P = \mathbb{CP}_3$ as explained, for example, in ref. [15]. The manifold M with its conformal structure can be recovered from its *twistor space* P. We shall assume that the reader is familiar with this construction and with the two-spinor formalism [14] with which it is usually described.

For each $x \in P$ one has the corresponding rational curve $L_x \equiv \nu(\mu^{-1}(x))$ in M and hence a series of fattenings coming from its embedding in M. One can use the corollary above (with $p = 1$) to investigate this series. Any fattening $X^{(m)}$ may be compared with the trivial fattening obtained by embedding X as the zero section of its normal bundle. The deviation of $X^{(m)}$ from the trivial may be measured step by step in the cohomology $H^1(X, \mathcal{O}(\hat{T} \otimes \odot^m N^*))$ but only the first non-vanishing such class will be well defined. To compute this cohomology in our example it is convenient to work on F. As x varies, the normal bundles to L_x fit together to yield the normal bundle to F in $P \times M$, namely $N = \mathcal{O}^A(1)$ following the notation of ref. [5] adapted to this more general setting. Moreover, the tangent bundle $\hat{T}$ pulls back onto F to give an exact sequence

$$0 \to \mathcal{O}(2)[-1] \to \mu^*\hat{T} \to N \to 0,$$

where square brackets denote conformal weight. One has

$$\mathcal{O}(2)[-1] \otimes \odot^m N^* = \mathcal{O}_{\underbrace{(AB\cdots D)}_{m}}(2-m)[-1],$$

so $H^1(F, \mathcal{O}(2)[-1] \otimes \odot^m N^*) = 0$ for $m \leq 3$. Further,

$$N \otimes \odot^m N^* = \mathcal{O}_{\underbrace{(AB\cdots C)}_{m-1}}(1-m) \oplus \mathcal{O}_{\underbrace{(AB\cdots E)}_{m+1}}(1-m)[1],$$

so $H^1(F, \mu^*\hat{T} \otimes \odot^m N^*) = 0$ for $m \leq 2$ and the first possibly non-vanishing class therefore lies in

$$\begin{aligned} H^1(F, \mu^*\hat{T} \otimes \odot^3 N^*) &= H^1(F, \mathcal{O}_{(AB)}(-2)) \oplus H^1(F, \mathcal{O}_{(ABCD)}(-2)[1]) \\ &= \Gamma(M, \mathcal{O}_{(AB)}[-1]) \oplus \Gamma(M, \mathcal{O}_{(ABCD)}). \end{aligned}$$

In particular, $F^{(2)}$ may be trivialized. The particular choice of trivialization, however, is acted upon by $H^0(F,\mu^*\hat{T}\otimes\bigodot^2 N^*) = \Gamma(M,\mathcal{O}_{(AB)}[-1])$. We claim that this choice acts on $H^1(F,\mu^*\hat{T}\otimes\bigodot^3 N^*)$ by translation in the first summand leaving precisely $\Gamma(M,\mathcal{O}_{(ABCD)})$ as governing the third order fattening and that the field so obtained is precisely the anti-self-dual Weyl curvature $\tilde{\Psi}_{ABCD}$. Notice that $\mathcal{O}_{(AB)}[-1]$ is precisely the bundle of anti-self-dual two-forms on M. Our claim is therefore initially made on the grounds of naturality—the Weyl curvature is expected as the first reasonable conformal invariant whereas there is no invariantly defined anti-self-dual two-form on M.

In order to prove this, let us compute the Penrose transform of $H^1(P,TP)$ continuing on from the general discussion in section 5 (see also ref. [1]). Using standard spinor notation [14] we have:

$$\begin{aligned} \mu^*TP &= \mathcal{O}_A(1)[1]+\mathcal{O}(2)[-1],\\ \Omega^1_\mu(\mu^*TP) &= \mathcal{O}_A(1)[-1]\otimes\mu^*TP = \mathcal{O}_{AB}(2)+\mathcal{O}_A(3)[-2],\\ \Omega^2_\mu(\mu^*TP) &= \mathcal{O}(2)[-3]\otimes\mu^*TP = \mathcal{O}_A(3)[-2]+\mathcal{O}(4)[-4], \end{aligned}$$

and therefore

$$\begin{aligned} \mathcal{E}^0 &= \nu_*(\mu^*TP) = \mathcal{O}_{AA'}[2]+\mathcal{O}_{(A'B')}[1],\\ \mathcal{E}^1 &= \nu_*(\Omega^1_\mu(\mu^*TP)) = \mathcal{O}_{AB(A'B')}[2]+\mathcal{O}_{A(A'B'C')}[1],\\ \mathcal{E}^2 &= \nu_*(\Omega^2_\mu(\mu^*TP)) = \mathcal{O}_{A(A'B'C')}[1]+\mathcal{O}_{(A'B'C'D')}. \end{aligned}$$

The cohomology of the complex $\Gamma(M,\mathcal{E}^\bullet)$ is the same as the cohomology of

$$\Gamma(M,\mathcal{O}_{AA'}[2])\to\Gamma(M,\mathcal{O}_{(AB)(A'B')}[2])\to\Gamma(M,\mathcal{O}_{(A'B'C'D')})$$

where the maps are given by

$$\begin{aligned} k_A^{A'} &\mapsto \nabla^{(A'}_{(A}k^{B')}_{B)},\\ h_{ABA'B'} &\mapsto \nabla^A_{(A'}\nabla^B_{B'}h_{C'D')AB}+\Phi^{AB}_{(A'B'}h_{C'D')AB}. \end{aligned}$$

Here, ∇_a is the metric connection and $-2\Phi_{ab}$ its trace-free Ricci curvature following the conventions of ref. [14].

We claim that $H^1(P,TP)$ is represented by h_{ab} as an infinitesimal change in conformal metric. The discussion of section 4 is precisely what is needed to investigate this claim—in the light of the Penrose transform constructed in section 5, theorem 5 may be interpreted as saying that the Kodaira–Spencer map is inverse to the Penrose transform. So, let us start with an infinitesimal deformation of the conformal metric and see how this gives rise to a Kodaira–Spencer element of $H^1(P,TP)$.

Fix a metric g_{ab} in the conformal class on M. Let ∇_a denote the corresponding metric connection and also the connection on spinors as in ref. [14]. Recall [13] that F is the projective primed spin bundle over M. By horizontal lifting, ∇_a becomes a differential operator on all natural irreducible bundles $\mathcal{O}_{(AB\cdots D)}(k)[w]$ on F. Combining with the tautological section $\pi^{A'} \in \Gamma(F, \mathcal{O}^{A'}(1)[-1])$ gives a differential operator $\nabla_A \equiv \pi^{A'}\nabla_{AA'}$ which, acting on functions, defines a rank-two distribution. Using formulae for spinor connections derived in ref. [14], it follows that

$$\nabla_A\nabla^A f = \pi^{A'}\pi^{B'}\pi_{C'}\Box_{A'B'}\partial^{C'} f = -\pi^{A'}\pi^{B'}\pi^{C'}\tilde{\Psi}_{A'B'C'D'}\partial^{D'} f,$$

where $\partial^{D'} f = \partial f/\partial\pi_{D'}$ is the natural differential operator $\mathcal{O}(k) \to \mathcal{O}^{D'}(k-1)$ along the fibres of ν. Thus, the distribution is integrable if and only if $\tilde{\Psi}_{A'B'C'D'} = 0$. In other words, M is conformally anti-self-dual. In this case, P is the space of leaves of this integrable distribution.

We wish to deform the metric g_{ab} but at the same time keep F fixed so as to be able to apply theorem 5. The way to do this (following a similar manœuvre used for ambitwistors in ref. [10]) is to consider metrics induced by automorphisms of the tangent bundle as follows. Let $\phi_a{}^b : TM \to TM$ denote an automorphism of the tangent bundle. Such an automorphism gives rise to a new metric $\tilde{g} = (\phi^{-1})^* g$. Thus, $\phi_a{}^c\phi_b{}^d\tilde{g}_{cd} = g_{ab}$ and $\tilde{g}_{ab}$ is characterized by $\tilde{g}_{ab}\tilde{X}^a\tilde{X}^b = g_{ab}X^aX^b$ where $\tilde{X}^a = X^b\phi_b{}^a$. The automorphism ϕ may be used to carry spinors for g_{ab} into spinors for $\tilde{g}_{ab}$ and so F may be regarded as fixed. An infinitesimal automorphism has the form $\phi_a{}^b = \delta_a{}^b + tX_a{}^b$ modulo terms of order t^2. From now on we shall neglect terms of order t^2 in all computations. Thus,

$$\tilde{g}_{ab} = (\delta_a{}^c - tX_a{}^c)(\delta_b{}^d - tX_c{}^d)g_{cd} = g_{ab} - t(X_{ab} + X_{ba}).$$

Hence, we may as well assume that X_{ab} is symmetric and, since we are only interested in conformal metrics (it is easy to check that a conformal change of metric has no effect on the twistor construction), we may also assume that X_{ab} is trace free. Let $h_{ab} = -\frac{1}{2}X_{ab}$ so that $\tilde{g}_{ab} = g_{ab} + th_{ab}$. The requirement that h_{ab} be trace free symmetric implies that $h_{ABA'B'} = h_{(AB)(A'B')}$. The change in connection is given by

$$\tilde{\nabla}_a\omega_b = \nabla_a\omega_b + \tfrac{1}{2}t[\nabla^c h_{ab} - \nabla_a h_b{}^c - \nabla_b h_a{}^c]\omega_c,$$

but, in order to view this as a change of differential operator keeping TM fixed, we must intertwine it with the automorphism $\phi_a{}^b$ to form

$$\begin{aligned} D_a\omega_b &\equiv \phi_a{}^c\phi_b{}^d\tilde{\nabla}_c((\phi^{-1})_d{}^e\omega_e) \\ &= (\delta_a{}^c - \tfrac{1}{2}th_a{}^c)(\delta_b{}^d - \tfrac{1}{2}th_b{}^d)\tilde{\nabla}_c((\delta_d{}^e + \tfrac{1}{2}th_d{}^e)\omega_e) \\ &= \nabla_a\omega_b - \tfrac{1}{2}th_a{}^c\nabla_c\omega_b + \tfrac{1}{2}t[\nabla^c h_{ab} - \nabla_b h_a{}^c]\omega_c. \end{aligned}$$

D_a is not strictly a connection but rather satisfies a Leibniz rule

$$D_a(f\omega_b) = fD_a\omega_b + (D_af)\omega_b,$$

where $D_af = \nabla_a f - \frac{1}{2}th_a{}^c\nabla_c f$ is a modified exterior derivative. With this proviso, D_a extends to an operator on all tensors and is metric preserving and torsion free in the sense that

$$D_a g_{bc} = 0 \quad \text{and} \quad D_{[a}D_{b]}f = 0.$$

By general theory, D_a should extend to an operator on spinors and indeed

$$\nabla_c h_{ab} - \nabla_b h_{ac} = \epsilon_{BC}\nabla^D_{(B'}h_{C')AA'D} + \epsilon_{B'C'}\nabla^{D'}_{(B}h_{C)AA'D'},$$

so we find that

$$D_{AA'}\omega_B = \nabla_{AA'}\omega_B - \tfrac{1}{2}th_{AA'}{}^{CC'}\nabla_{CC'}\omega_B - \tfrac{1}{2}t[\nabla^{D'}_{(B}h_{C)AA'D'}]\omega^C,$$

$$D_{AA'}\omega_{B'} = \nabla_{AA'}\omega_{B'} - \tfrac{1}{2}th_{AA'}{}^{CC'}\nabla_{CC'}\omega_{B'} - \tfrac{1}{2}t[\nabla^D_{(B'}h_{C')AA'D}]\omega^{C'}.$$

We may lift D_a to operators on F. For example,

$$D_{AA'}f = \nabla_{AA'}f - \tfrac{1}{2}th_{AA'}{}^{CC'}\nabla_{CC'}f + \tfrac{1}{2}t\pi^{C'}[\nabla^D_{(B'}h_{C')AA'D}]\partial^{C'}f$$

acting on (homogeneous) functions. This allows us to define $D_A = \pi^{A'}D_{AA'}$ as a perturbation of the operator and distribution ∇_A. To see when D_A is integrable we compute

$$\begin{aligned}
D_AD^Af &= \nabla_A\nabla^Af \\
&\quad -\tfrac{1}{2}t\pi^{B'}h^A{}_{B'}{}^{DD'}\nabla_A\nabla_{DD'}f - \tfrac{1}{2}t\pi^{B'}h_{AB'}{}^{DD'}\nabla_{DD'}\nabla^Af \\
&\quad +\tfrac{1}{2}t\pi^{B'}\pi^{C'}[\nabla_A\nabla^D_{(C'}h_{D')B'}{}^A{}_D]\partial^{D'}f \\
&\quad +\tfrac{1}{2}t\pi^{B'}\pi^{C'}[\nabla^D_{(C'}h_{D')B'}{}^A{}_D]\nabla_A\partial^{D'}f \\
&\quad +\tfrac{1}{2}t\pi^{A'}\pi^{C'}[\nabla^D_{(B'}h_{C')A'AD}]\partial^{B'}\nabla^Af \\
&\quad -\tfrac{1}{2}t\pi^{B'}[\nabla_Ah^A{}_{B'}{}^{DD'}]\nabla_{DD'}f - \tfrac{1}{4}t\pi^{A'}[\nabla^{AD'}h_{CAA'D'}]\nabla^Cf \\
&= \nabla_A\nabla^Af + \tfrac{1}{2}t\pi^{A'}\pi^{B'}\{h^{AD}{}_{B'A'}\pi_{C'}\Box_{AD}\partial^{C'}f \\
&\quad -\pi^{C'}[\nabla^A_{A'}\nabla^D_{(C'}h_{D')B'AD}]\partial^{D'}f\} \\
&\quad -\tfrac{1}{2}t\pi^{A'}\pi^{C'}\{[\nabla^D_{(B'}h_{C')A'AD}]\nabla^{AB'}f \\
&\quad +[\nabla_{AA'}h^A{}_{C'}{}^{DD'}]\nabla_{DD'}f + \tfrac{1}{2}t[\nabla^{AD'}h_{CAA'D'}]\nabla^C_{C'}f\} \\
&= \nabla_A\nabla^Af - \tfrac{1}{2}t\pi^{A'}\pi^{B'}\pi^{C'}\{\nabla^A_{A'}\nabla^D_{(C'}h_{D')B'AD} + h^{AD}_{A'B'}\Phi_{C'D'AD}\}\partial^{D'}f \\
&= -\pi^{A'}\pi^{B'}\pi^{C'}\{\tilde{\Psi}_{A'B'C'D'} + \tfrac{1}{2}t[\nabla^A_{(A'}\nabla^B_{B'}h_{C'D')AB} + \Phi^{AB}_{(A'B'}h_{C'D')AB}]\}\partial^{D'}f.
\end{aligned}$$

Recall that $\tilde{\Psi}_{A'B'C'D'} = 0$. Thus, the perturbed distribution is integrable if and only if

$$\nabla^{AB}_{(A'B'}h_{C'D')AB} + \Phi^{AB}_{(A'B'}h_{C'D')AB} = 0.$$

Notice for later use that a similar calculation on the unprimed spin bundle shows that the anti-self-dual curvature of the perturbed metric $\tilde{g}_{ab}$ written in terms of spinors for g_{ab} is

$$\Psi_{ABCD} + \tfrac{1}{2}t[\nabla^{A'}_{(A}\nabla^{B'}_{B}h_{CD)A'B'} + \Phi^{A'B'}_{(AB}h_{CD)A'B'}].$$

To relate this infinitesimal change in distribution to its Kodaira–Spencer image in $H^1(P, TP)$ as in section 4, we must view it as a section of $\Omega^1_\mu(\mu^*TP)$. Whilst the above calculations are valid for any homogeneous f, if the homogeneity is zero, then $\partial^{C'}f = \pi^{c'}\eth f$, where $\eth : \mathcal{O} \to \mathcal{O}(-2)[1]$ is the exterior derivative along the fibres of ν. Thus,

$$D_A f = \nabla_A f - \tfrac{1}{2}t\pi^{A'}h_{AA'}{}^{BB'}\nabla_{BB'}f + \tfrac{1}{2}t\pi^{A'}\pi^{B'}\pi^{C'}[\nabla^D_{A'}h_{B'C'AD}f]\eth f.$$

However, the operator $f \mapsto (\eth f, \nabla_b f) \in \mathcal{O}(-2)[1] + \mathcal{O}_b$ is just the exterior derivative d on F so the deformed relative exterior derivative $\tilde{d}_\mu = D_A$ is related to the original relative exterior derivative $d_\mu = \nabla_A$ by

$$\tilde{d}_\mu = d_\mu - \tfrac{1}{2}t\psi \lrcorner d,$$

where $\psi \in \Omega^1_\mu(TF) = \mathcal{O}_A(1)[-1] \otimes (\mathcal{O}_{BB'}[2] + \mathcal{O}(2)[-1])$ is given by

$$\begin{aligned}\psi &= (\pi^{A'}h_{AA'BB'}, -\pi^{A'}\pi^{B'}\pi^{C'}\nabla^D_{A'}h_{B'C'AD}) \\ &\in \mathcal{O}_{ABB'}(1)[1] + \mathcal{O}_A(3)[-2].\end{aligned}$$

The natural quotient mapping to $\Omega^1_\mu(\mu^*TP)$ is defined by contraction with $\pi^{B'}$ in the first factor giving

$$\begin{aligned}\phi &= (\pi^{A'}\pi^{B'}h_{A'B'AB}, -\pi^{A'}\pi^{B'}\pi^{C'}\nabla^C_{A'}h_{B'C'AD}) \\ &\in \mathcal{O}_{AB}(2) + \mathcal{O}_A(3)[-2].\end{aligned}$$

In particular, the direct image in $\Gamma(M, \nu_*\mathcal{O}_{AB}(2)) = \Gamma(M, \mathcal{O}_{AB(A'B')}[2])$ gives h_{ab} as required.

We remark that a more complete discussion of the Penrose transform from this point of view would necessitate a further investigation of the differentials of the resolution

$$\mu^*TP \to \Omega^1_\mu(\mu^*TP) \to \Omega^2_\mu(\mu^*TP).$$

This would force us to consider *local twistors* and go into a considerable digression. We refer to ref. [11] as an indication of how to carry out such an investigation.

Now fix $x \in M$ and the corresponding line $X \subset P$. We have shown that $H^1(x, \hat{T} \otimes \bigodot^m N) = 0$ for $m \leq 2$ and so $X^{(2)}$ may be trivialized. Notice, however, that

$$\begin{aligned} H^0(X, \hat{T} \otimes \textstyle\bigodot^m N^*) &= \Gamma(X, (\mathcal{O}_C(1)[1] + \mathcal{O}(2)[-1]) \otimes \mathcal{O}_{(AB)}(-2)) \\ &= \Gamma(X, \mathcal{O}_{(AB)C}(-1)[1] + \mathcal{O}_{(AB)}[-1]) \\ &= \mathcal{O}_{(AB)}[-1], \end{aligned}$$

so there is this degree of freedom in how $X^{(2)}$ may be identified with the trivial fattening.

In any case, suppose we fix a trivialization of $X^{(2)}$. According to section 3, we may construct the difference between $X^{(3)}$ and the trivial extension by integrating the Kodaira–Spencer elements of $H^1(X, \hat{T} \otimes \bigodot^3 N^*)$ obtained as the images of the infinitesimal deformations preserving $X^{(2)}$, i.e. elements of $H^1(P, \mathcal{I}_X^3(TP))$. Thus, we must investigate the Penrose transform of this space.

The procedure is very similar to the transform of $H^1(P, TP)$ above except that we must take into account the vanishing to third order along X. Let X also denote the fibre $\nu^{-1}(x)$ above x in F. Vanishing along $X \subset F$ together with constancy along the fibres of μ is equivalent to the vanishing along $X \subset P$. Thus, it is easy to see that the sequence

$$\mathcal{I}_X^3(\mu^* TP) \to \mathcal{I}_X^2(\Omega_\mu^1(\mu^* TP)) \to \mathcal{I}_X(\Omega_\mu^2(\mu^* TP)) \to 0$$

is a resolution of $\mu^{-1}(\mathcal{I}_X^3(TP))$ on F. Taking direct images under ν gives

$$\begin{aligned} \mathcal{F}^0 &= \mathcal{I}_x^3 \otimes (\mathcal{O}_{AA'}[2] + \mathcal{O}_{(A'B')}[1]), \\ \mathcal{F}^1 &= \mathcal{I}_x^2 \otimes (\mathcal{O}_{AB(A'B')}[2] + \mathcal{O}_{A(A'B'C')}[1]), \\ \mathcal{F}^2 &= \mathcal{I}_x \otimes (\mathcal{O}_{A(A'B'C')}[1] + \mathcal{O}_{(A'B'C'D')}), \end{aligned}$$

such that the cohomology $\Gamma(M, \mathcal{F}^\bullet)$ is the Penrose transform of $H^r(P, TP)$. This is slightly more difficult to analyze, but first note that the cokernel of

$$\mathcal{I}_x^3 \otimes \mathcal{O}_{(A'B')}[1] \to \mathcal{I}_x^2 \otimes \mathcal{O}_{AB(A'B')}[2]$$

is

$$\begin{aligned} &(\mathcal{I}_x^2/\mathcal{I}_x^3 \otimes \mathcal{O}_{(A'B')}[1]) \oplus (\mathcal{I}_x^2 \otimes \mathcal{O}_{(AB)(A'B')}[2]) \\ &= \mathbb{C}_{(AB)(A'B')(C'D')}[1] \oplus \mathbb{C}_{(A'B')}[-1] \oplus (\mathcal{I}_x^2 \otimes \mathcal{O}_{(AB)(A'B')}[2]), \end{aligned}$$

where $\mathbb{C}_A$ and $\mathbb{C}_{A'}$ denote the two spin spaces at $x \in M$, and that the cokernel of

$$\mathcal{I}_x^2 \otimes \mathcal{O}_{A(A'B'C')}[1] \to \mathcal{I}_x \otimes \mathcal{O}_{A(A'B'C')}[1]$$

is

$$\mathbb{C}_{AA'} \otimes \mathbb{C}_{B(B'C'D')}[1] = \mathbb{C}_{AB(A'B'C'D')}[1] \oplus \mathbb{C}_{AB(A'B')}.$$

There is an induced mapping

$$\mathbb{C}_{(AB)(A'B')(C'D')}[1] \oplus \mathbb{C}_{(A'B')}[-1] \to \mathbb{C}_{AB(A'B'C'D')}[1] \oplus \mathbb{C}_{AB(A'B')},$$

whose kernel is $\mathbb{C}_{(AB)}[-1]$ and whose cokernel is $\mathbb{C}_{(A'B'C'D')}$. Consequently, the cohomology of $\Gamma(M, \mathcal{F}^\bullet)$ is the same as the cohomology of

$$\Gamma(\mathcal{I}_x^3 \otimes \mathcal{O}_{AA'}[2]) \to \Gamma(\mathcal{I}_x^2 \otimes \mathcal{O}_{(AB)(A'B')}) \oplus \mathbb{C}_{(AB)}[-1] \to \Gamma(\mathcal{O}_{(A'B'C'D')}),$$

where the differential operators are exactly as before. Thus, the Penrose transform gives

$$H^1(P, \mathcal{I}_X^3(TP)) = \mathbb{C}_{(AB)}[-1] \oplus D,$$

where D is the space

$$\frac{\left\{\begin{array}{c} h_{ab} \in \Gamma(M, \mathcal{O}_{(AB)(A'B')}[2]) \\ \text{s.t. } \nabla^A_{(A'}\nabla^B_{B'}h_{C'D')AB} + \Phi^{AB}_{(A'B'}h_{C'D')AB} = 0 \\ h_{ab}|_x = 0 \;\; \nabla_a h_{bc}|_x = 0 \end{array}\right\}}{\left\{\begin{array}{c} h_{AB}^{A'B'} = \nabla^{A'}_{(A}k^{B'}_{B)} \text{ for some } k_b \in \Gamma(M, \mathcal{O}_b[2]) \\ \text{s.t. } k_b|_x = 0 \;\; \nabla_a k_b|_x = 0 \;\; \nabla_a\nabla_b k_c|_x = 0 \end{array}\right\}}.$$

As before, h_{ab} represents an infinitesimal change in conformal metric but now the gauge fixing at x means that a corresponding change in anti-self-dual Weyl curvature, namely $\nabla^{A'}_{(A}\nabla^{B'}_{B}h_{CD)A'B'}|_x$, makes good sense.

It remains to show that this infinitesimal change of curvature may be interpreted precisely as the second component of the restriction mapping

$$H^1(P, \mathcal{I}_X^3(TP)) \to H^1(X, \hat{T} \otimes \bigodot^3 N^*) = \mathbb{C}_{(AB)}[-1] \oplus \mathbb{C}_{(ABCD)}.$$

To see this we can relate resolutions on F

$$\begin{array}{ccccc} \mathcal{I}_X^3(\mu^*TP) & \longrightarrow & \mathcal{I}_X^2(\Omega^1_\mu(\mu^*TP)) & \longrightarrow & \mathcal{I}_X(\Omega^2_\mu(\mu^*TP)) \\ \downarrow & & \downarrow & & \downarrow \\ \bigodot^3\mathbb{C}_{AA'} \otimes (\mu^*TP)_X & \to & \bigodot^2\mathbb{C}_{AA'} \otimes (\Omega^1_\mu(\mu^*TP))_X & \to & \mathbb{C}_{AA'} \otimes (\Omega^2_\mu(\mu^*TP))_X \end{array}$$

noting that the second sequence resolves $\mu^{-1}(\hat{T}\otimes\bigodot^3 N^*)$. Taking direct images of this second sequence gives

$$\begin{array}{c}(\mathbb{C}_{(ABC)(A'B'C')}\oplus\mathbb{C}_{AA'}[-2])\otimes(\mathbb{C}_{DD'}[2]\oplus\mathbb{C}_{(D'E')}[1])\\ \downarrow\\ (\mathbb{C}_{(AB)(A'B')}\oplus\mathbb{C}[-2])\otimes(\mathbb{C}_{CD(C'D')}[2]\oplus\mathbb{C}_{C(C'D'E')}[1])\\ \downarrow\\ \mathbb{C}_{AA'}\otimes(\mathbb{C}_{B(B'C'D')}[1]\oplus\mathbb{C}_{(B'C'D'E')})\end{array}$$

which has cohomology only in the middle position providing an alternative derivation that

$$H^1(x,\hat{T}\otimes\bigodot^3 N^*) = \mathbb{C}_{(AB)}[-1]\oplus\mathbb{C}_{(ABCD)}.$$

This also shows that

$$\begin{array}{ccc}H^1(P,\mathcal{I}^3_X(TP)) & \longrightarrow & H^1(X,\hat{T}\otimes\bigodot^3 N^*)\\ \| & & \|\\ \mathbb{C}_{(AB)}[-1] & \overset{=}{\longrightarrow} & \mathbb{C}_{(AB)}[-1]\\ \oplus & & \oplus\\ D & \longrightarrow & \mathbb{C}_{(ABCD)}\end{array}$$

is given by $h_{ab}\mapsto\nabla^{A'}_{(A}\nabla^{B'}_{B}h_{CD)A'B'}|_x$ as required. It remains to interpret the component of $H^1(P,\mathcal{I}^3_X(TP))$ or $H^1(X,\hat{T}\otimes\bigodot^3 N^*)$ in $\mathbb{C}_{(AB)}[-1]$. This is an artifact of the chosen trivialization as reflected in the isomorphism

$$H^0(X,\hat{T}\otimes\bigodot^2 N^*) = \mathbb{C}_{(AB)}[-1].$$

Integrating up as in section 3 proves the following:

Theorem 7. *There is a preferred trivialization of $X^{(2)}$ such that the difference between $X^{(3)}$ and the trivial fattening in*

$$H^1(X,\hat{T}\otimes\bigodot^3 N^*) = \mathbb{C}_{(AB)}[-1]\oplus\mathbb{C}_{(ABCD)}$$

has first component equal to zero. The second component is precisely the anti-self-dual Weyl curvature at $x\in M$.

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